# Market Price of Risk Analysis from Three Major Industrial Countries on the Stability of the Brennan-Schwartz Model

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# INDONESIAN
# CAPITAL MARKET REVIEW

# Market Price of Risk Analysis from Three Major Industrial Countries on the Stability of the Brennan-Schwartz Model

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*At any given time, market price of risk must be the same for all derivatives and it is linked in particular to interest rate. The Brennan-Schwartz model is one of the stochastic differential equations for the interest rate under the risk neutral probability measure. To estimate parameters of this model, it is required that the real data which are collected in the real world in which the distribution of interest rate process is under the actual probability measure. Therefore, parameter estimators are obtained by changing the measure which is determined by the market price of risk. Hence, market price of risk must make the Brennan-Schwartz model becomes stable, which is important to describe resistance of the model to the perturbation in the initial state or parameters of the model. This paper aims to analyze the market price of risk from three major industrial countries: USA, Japan, and Canada. This analysis can be used as a guideline to decide that the interest rate of these three major industrial countries can be modeled as Brennan-Schwartz model.*

*Keywords:* Market price of risk, Brennan-Schwartz model, stability

## Introduction

Pennacchi (2008) defines that derivatives are securities whose value is dependent upon outcome of other variables because their cashflows derive from another underlying variables, such as an asset price, interest rate, or exchange rate. For example, a European call option on a zero-coupon bond that does not only depends on the underlying zero-coupon bond price but also on the short-rate which have to be considered stochastic (current and future) (Murni and Handhika, 2011). In the modern asset pricing theory, risk neutral valuation is a tool for pricing derivatives based on the risk neutral world $\tilde{P}$. However, data are collected in the real world and their statistical properties characterize the distribution of interest rate process under the actual probability measure $P$ (Brigo and Mercurio, 2006). It is plausible to assume that, in

the real world, the return that an investor will, in general, demand from a security should be a function not only of its expected return, but also of the uncertainty connected with it (Rebonato, 1998). The precise return will depend on the investor's appetite for risk such as risk averse, risk seeker, or risk neutral. Therefore, the fair price of the derivatives, in the sense that unlimited profits could be made, can be recovered by making use of the real world probabilities, but only if these are used in conjunction with the market price of risk relating to the risk neutral world.

Hull (2003) defines market price of risk as a measure of the tradeoffs investors make between risk and return. It is security independent, which can be interpreted as an extra compensation (per unit of risk) for taking on risk. Since a market price of risk cannot depend on any specific feature of the particular security used to

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obtain it, it must also hold true in general for any security (Rebonato, 1998). In other words, at any given time, market price of risk must be the same for all derivatives and it is linked in particular to interest rate.

For any given currency, many different types of interest rates are regularly quoted and, in the real world, its dynamic is expressed as a stochastic differential equation. It depends on the parameter which contributes to the market price of risk. Shifting to the risk neutral world eliminates extra return for accepting risk. Therefore, most interest rate processes are based on the risk neutral interest rate dynamics, e.g. the Brennan-Schwartz model, which drift term already includes adjustment for market price of interest rate risk (Aling and Hassan, 2012; Brennan and Schwartz, 1980; and Handhika, 2011).

The Brennan-Schwartz model requires estimation of parameters whose values are unknown. Since data are collected in the real world, we need to change the measure of the Brennan-Schwartz model under the risk neutral probability measure $\tilde{P}$ to the Brennan-Schwartz model under the actual probability measure $v$ that depends on the new parameter which contributing to the market price of risk. Market price of risk must make the Brennan-Schwartz model becomes stable, which is important to describe resistance of the model to the perturbation in the initial state or parameters of the model. This paper aims to analyze the market price of risk from three major industrial countries: USA, Japan, and Canada. This analysis can be used as a guideline to decide that the interest rate of these three major industrial countries can be modeled as Brennan-Schwartz model.

## Literature Review

Models for the interest rate, $r_t$, are sometimes called short rate models because $r_t$ is the interest rate for short term borrowing. When the interest rate is determined by only one stochastic differential equation, as is the case in this paper, the model is said to have one factor. The primary shortcoming of one factor models is that they cannot capture complicated yield curve behavior and they tend to produce paral-

lel shifts in the yield curve but not changes in its slope or curvature (Shreve, 2004).

Brennan and Schwartz (1980) introduce the Brennan-Schwartz model as one of the one-factor stochastic differential equation models for the short rate under the risk neutral probability measure $\tilde{P}$ as follows:

$$dr_t = \gamma\left(\alpha - r_t\right)dt + \beta r_t d\tilde{W}_t, \qquad (1)$$

where $\gamma$, $\alpha$, $\beta$ are nonnegative constants and $\tilde{W}_t$ is a Brownian motion under the risk neutral probability measure $\tilde{P}$. $\alpha$ is defined as a reversion level of the rate of interest $r_t$, $\gamma$ as a speed of adjustment, and $\beta^2 r_t^2$ as the infinitesimal variance of the process. The quantity $\gamma\left(\alpha - r_t\right)$ and $\beta r_t$ are called the drift and diffusion coefficients of the process, respectively. It is seen that the drift coefficient depends on the current value of $r_t$. If $r_t$ is less than $\alpha$, then the drift coefficient is positive; if $r_t$ is greater than $\alpha$, then it is negative. Thus, the interest rate $r_t$ tends to revert to its reversion level $\alpha$ at a rate which depends upon the speed of adjustment $\gamma$ (Brennan and Schwartz, 1980).

Like Vasicek, Rendleman-Bartter, or Cox-Ingersoll-Ross model, the Brennan-Schwartz model also belongs to the equilibrium model that has the properties of mean reversion. It is similar to the Vasicek model, but the diffusion coefficient is multiplicative. Hull (2003) illustrates mean reversion property as in Figure 1.

Handhika (2011) shows that the Brennan-Schwartz model has a unique solution based on the existence and the uniqueness theorem (Öksendal, 1998) and it can be solved explicitly by applying Ito-Doeblin formula and Ito product rule (Mikosch, 1998 and Shreve, 2004) as follows:

$$r_t = r_0 \cdot \exp\left\{-\left(\gamma + \frac{1}{2}\beta^2\right)t + \beta\tilde{W}_t\right\} +$$

$$\gamma\alpha\int_0^t \exp\left\{-\left(\gamma + \frac{1}{2}\beta^2\right)(t-u) +\right.$$

$$\left.\beta\left(\tilde{W}_t - \tilde{W}_u\right)\right\} du. \qquad (2)$$

From the existence and the uniqueness theory we know that the solutions of a differential
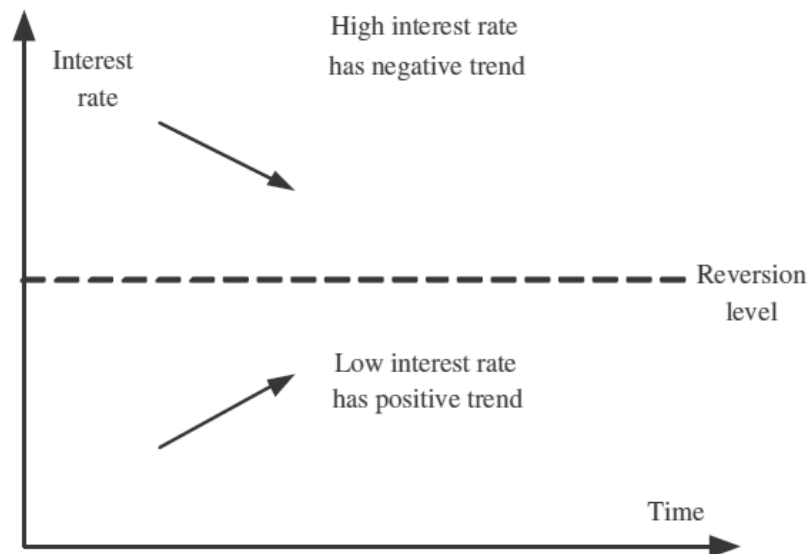
Figure 1. Mean reversion property

equation are continuous in their initial values, at least over a finite time interval. Extending this idea to an infinite time interval leads to the concept of stability (Kloeden and Platen, 1992). Stability is important to describe resistance of the model to the perturbation in the initial state or parameters of the model (Arnold, 1974). Two ways to define stochastic stability would be considered in this paper, stochastically asymptotically stable and mean-square stable. Handhika (2011) finds that their criteria for the Brennan-Schwartz model are $\gamma + \frac{1}{2}\beta^2 > 0$ and $2\gamma - \beta^2 > 0$, respectively. These stability criteria can be used as guidelines for selecting parameters that make the model becomes resistant to the perturbation.

## Research Method

In fact, the values of parameters of the Brennan-Schwartz model as in (1) are unknown. They need to be estimated by data which are collected in the real world. Actually, by using any parameter that is included in each stability criteria, it would be obtained a stable model, both stochastic asymptotic as well as mean square. However, it has not been able to describe the real problem being addressed related to the real data. Therefore, in this section, parameter estimation using data which are collected in the

real world would be discussed. Notice that their statistical properties characterize the distribution of interest rate process under the actual probability measure $P$, whereas the Brennan-Schwartz model as in (1) is under the risk neutral probability measure $\tilde{P}$. In other words, it is required the change of measure to the Brennan-Schwartz model when estimating its parameter by using Girsanov's Theorem and the previous trick method (Zeytun and Gupta, 2007) to simplify the estimation problem as shown in Handhika (2011). Finally, the Brennan-Schwartz model as in (1) can be represented as follows:

$$dr_t = \gamma^* \left( \alpha^* - r_t \right) dt + \beta^* r_t dW_t, \qquad (3)$$

where $\beta = \beta^*$, $\gamma = \gamma^* + \lambda\beta$, $\alpha = \dfrac{\gamma^* \alpha^*}{\gamma}$, $W_t$ is a Brownian motion under the actual probability measure $P$, and parameter $\lambda$ is defined as a market price of risk.

In this paper, the Maximum Likelihood estimation method would be implemented to estimate the parameters of the Brennan-Schwartz model under the actual probability measure $P$ as in (3) which depend on the market price of risk ($\lambda$). While the Brennan-Schwartz model as in (3) is formulated in continuous time, the sample data are always collected at discrete points in time or over discrete intervals in the case of flow data. Handhika (2011) shows that

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Table 1. Parameter estimators of the Brennan-Schwartz model under the actual probability measure $P$ as in (3) for the three major industrial countries: USA, Japan, and Canada

| Country | Parameter estimator | | |
| --- | --- | --- | --- |
| | $\hat{\gamma}^*$ | $\hat{\alpha}^*$ | $\hat{\beta}^*$ |
| USA | 0.2333 | 0.0317 | 0.7167 |
| Japan | 0.5660 | 0.0056 | 0.6096 |
| Canada | 1.1580 | 0.0314 | 0.2160 |

Table 2. Market price of risk criteria for the Brennan-Schwartz model under the risk neutral probability measure $\tilde{P}$ as in (1) for the three major industrial countries: USA, Japan, and Canada

| Country | Stability criteria | |
| --- | --- | --- |
| | Stochastically asymptotically stable | Mean-square stable |
| USA | $\lambda > - 0.68387$ | $\lambda > 0.03283$ |
| Japan | $\lambda > - 1.23328$ | $\lambda > - 0.62368$ |
| Canada | $\lambda > - 5.46911$ | $\lambda > - 5.25311$ |

the transition density of the Brennan-Schwartz model does not have a closed form analytic expression. To address this complication, Allen (2007) develops the Euler difference scheme approach involves approximating the likelihood function. Therefore, Handhika (2011) have also found the approximate log-likelihood function of (3) as follows:

$$\ell\left(\gamma^*,\alpha^*,\beta^*;t_k,r_k\,|\,t_{k-1},r_{k-1};k=1,2,\ldots,N\right)=$$

$$\ln\left(f_0\left(r_0\,|\,\gamma^*,\alpha^*,\beta^*\right)\right)$$

$$-\sum_{k=1}^{N}\left[\frac{\left(r_k-\left\{r_{k-1}+\gamma^*\left(\alpha^*-r_{k-1}\right)\Delta t\right\}\right)}{2\left(\beta^*\right)^2 r_{k-1}^2\Delta t}\right.$$

$$\left.+\ln\left(\beta^* r_{k-1}\sqrt{2\pi\Delta t}\right)\right], \tag{4}$$

where $r_k=r_{t_k}$ at $t=t_k$ for $r_{t_0}$, $r_{t_1}$, $r_{t_2}$, ..., $r_{t_N}$ are observed values of $r_t$, $0 \leq t \leq T$, at the respective uniformly distributed times $t_k = k \cdot \Delta t = \dfrac{k}{N}T$ for $k = 0, 1, 2, ..., N$.

Phillips and Yu (2009) show some characteristics of Euler difference scheme. When sampling interval $\Delta t$ is small, the Euler difference scheme should provide a good approximation

to the exact discrete time model. However, when it is large, the Euler approximation can be poor. The Euler discretization offers a good approximation to the exact discrete time model for daily or higher frequencies but not for annual or lower frequencies. The advantages of the Euler method include the ease with which the likelihood function is obtained, the low computational cost, and the wide range of its applicability. The biggest problem with the procedure is that when sampling interval is fixed, the estimator is inconsistent (Lo, 1988).

However, equation (4) cannot be analytically solved. Therefore, in this paper, the estimation would be continued iteratively using the Nelder-Mead Algorithm as given in Rouah and Vainberg (2007) which minimizing the $-\ell\left(\gamma^*,\alpha^*,\beta^*\right)$.

## Result and Discussion

The Brennan-Schwartz model will be implemented to approximate the real data of three major industrial countries: USA, Japan, and Canada since January 2001 until April 2011 in which data is obtained from Bloomberg, by using parameter estimators which are obtained from the Maximum Likelihood procedure. In this pa-

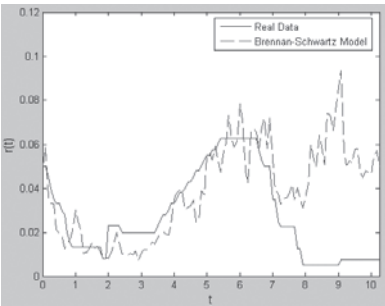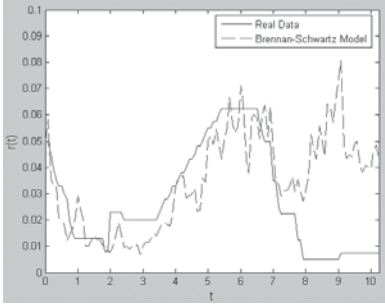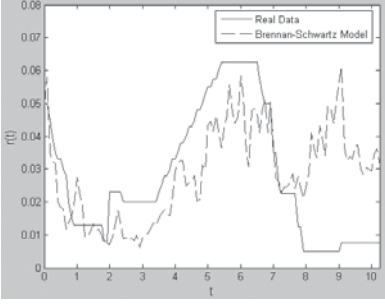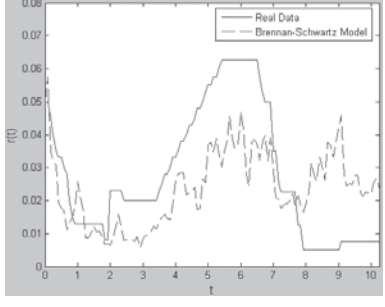Table 3. Interest rate's movements of the Brennan-Schwartz model for USA
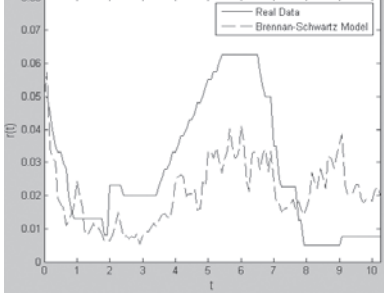
| Country | Market price of risk | Figure | Norm error |
|---------|---------------------|--------|-----------|
| | 0.05 |  | 8.61% |
| | 0.1 |  | 7.30% |
| USA | 0.2 |  | 5.36% |
| | 0.32 |  | 3.82% |
| | 0.4 |  | 4.12% |

Table 4. Interest rate's movements of the Brennan-Schwartz model for Japan
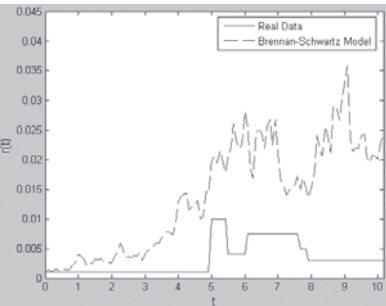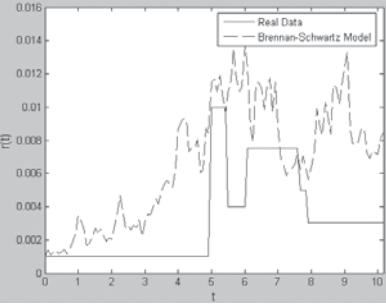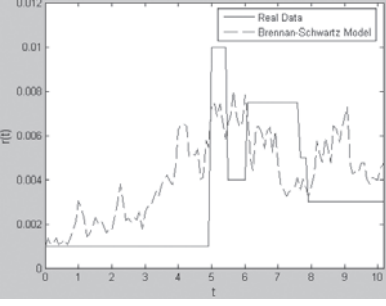
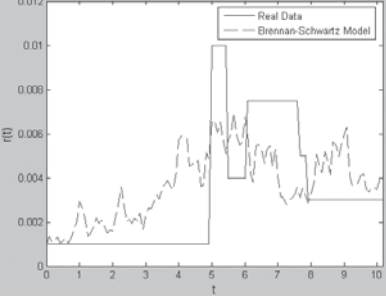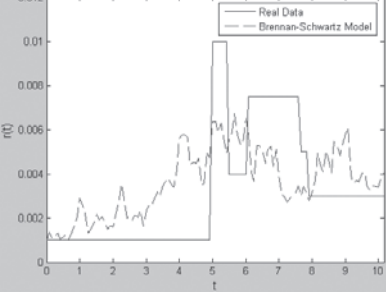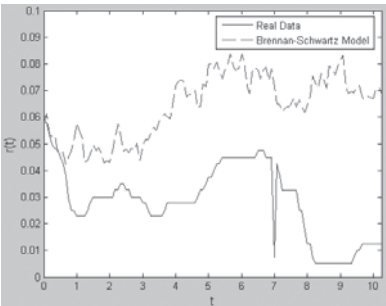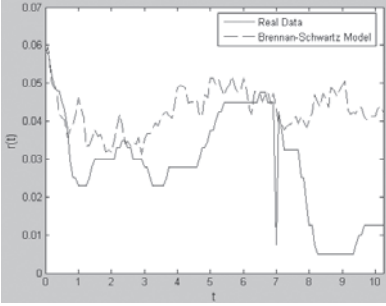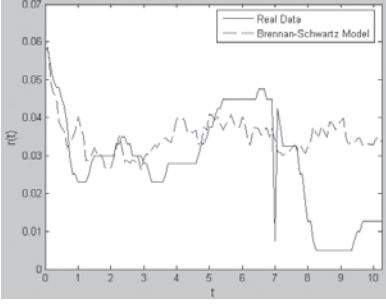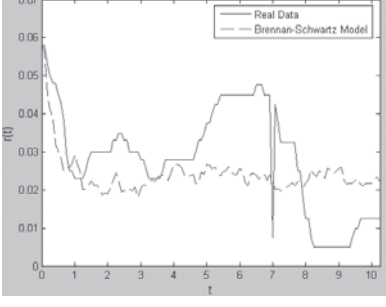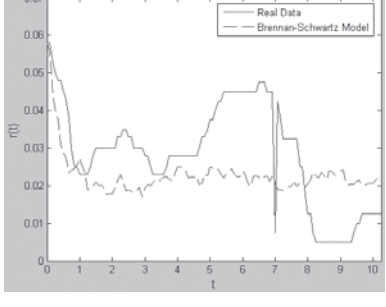| Country | Market price of risk | Figure | Norm error |
|---------|---------------------|--------|------------|
| | -0.50 |  | 3.29% |
| | 0.00 |  | 1.03% |
| Japan | 0.50 |  | 0.55% |
| | 0.66 |  | 0.49% |
| | 0.70 |  | 0.51% |

## Table 5. Interest rate's movements of the Brennan-Schwartz model for Canada

| Country | Market price of risk | Figure | Norm error |
|---------|---------------------|--------|------------|
| | -2.50 |  | 7.82% |
| | -1.00 |  | 4.56% |
| Canada | 0.00 |  | 3.49% |
| | 2.50 |  | 2.47% |
| | 3.00 |  | 2.62% |

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per, Matlab software is used to help calculation in the estimation problem and to illustrate the interest rate's movement in the approximation problem. Table 1 shows the parameter estimators of the Brennan-Schwartz model under the actual probability measure $P$ as in (3) for these three major industrial countries. Based on the results which have been shown in Table 1, the market price of risk criteria for parameter estimators which belong to the stochastically asymptotically and mean square stability criteria of the Brennan-Schwartz model, respectively, are shown in Table 2.

Figures in Table 3 - 5 visualized the approximation of interest rate's movements for the three major industrial countries: USA, Japan, and Canada, respectively, together with each norm error which is defined as the maximum absolute error. This visualization is the solution of the Brennan-Schwartz model (2) by using market prices of risk that make parameter estimators would belong to the both stability criteria of the Brennan-Schwartz model under the risk neutral probability measure $\tilde{P}$ as given in Table 2. Notice that in order to determine when market price of risk ($\lambda$) is positive or negative, one would require access to the utility function of the market investor, which specifies his attitude towards risk (Rebonato, 1998).

Figures in Table 3-5 show how interest rate's movements vary between market prices of risk that make a Brennan-Schwartz model becomes stable for each industrial country: USA, Japan, and Canada. In the context of interest rate models, if investors are risk-averse, then the market price of risk will be positive and they will demand an extra compensation from a risky security, on top of the return that he would earn, i.e. holding the riskless bond. If investors are risk neutral, then the market price of risk will be zero and they will accept a return exactly identical to the one obtainable from the bond. If, finally, investors are risk seekers, then the market price of risk will be negative and they will be happy with a lower return than the riskless rate (Rebonato, 1998, and Ahn and Shrestha, 2009).

Based on the results which have been shown in Table 2, for USA's interest rate, Brennan-Schwartz model is not a suitable model for in-

vestors who are classified as risk neutral and risk seeker because the market price of risk does not satisfy the mean square stability criteria of the Brennan-Schwartz model. In addition, Canada is a country with the widest stability criteria among the three industrial countries, followed by Japan and USA.

Figures in Table 3 show that the interest rate's movements of the Brennan-Schwartz model for USA is good enough in approximating the real data for the first eight years when each $\lambda$=0.05, $\lambda$=0.1, and $\lambda$=0.2. However, they give large norm error. Overall, for 10 years, $\lambda$=0.32 gives the smallest norm error among the five trial value of $\lambda$. Unlike the USA, Brennan-Schwartz model is less good in approximating the real interest rate's data of Japan and Canada. It might be caused by the existence of outliers as shown in Table 4 and 5 (the real data's movements). Although, for Japan, the norm error of the five trials value of $\lambda$ are relatively small but comparable with the real data which is small, i.e. around three decimal places.

In general, for these three industrial countries, it is seen that the interest rate's movements of the Brennan-Schwartz model by using parameter estimators which obtained from the Maximum Likelihood procedure is not really good enough in approximating the real data. It can be caused by several market prices of risk which are used to approximate the real data are based on trial and error. This analysis can be enhanced by combining the two approaches: Maximum Likelihood estimation method for estimating parameter of the Brennan-Schwartz model and the calibration technique for estimating the market price of risk as discussed in Brigo and Mercurio (2006) and Zeytun and Gupta (2007). Notice that market price of risk resulting by calibration technique should be included in the stability criteria of the model. Market prices of risk which are used in the analysis have been satisfied the stability criteria of Brennan-Schwartz model resulting in this paper. However, the concept of stability can be seen more clearly if the real data which is used in the analysis is not only around ten years, as used in this paper, but for long (infinite) time interval.

# Conclusion

In this paper, market price of risk as stability criteria of the Brennan-Schwartz model has been successfully derived by changing the measure and implementing the Maximum Likelihood procedure. Furthermore, the results can be used as a guideline to decide that the interest rate problems can be modeled as Brennan-Schwartz model. It had been tried to analyze the interest rate of three major industrial countries: USA, Japan, and Canada. Canada is a country with the widest stability criteria among the three industrial countries, followed by Japan and USA. In general, for these three industrial countries, it is seen that the interest rate's movements of the Brennan-Schwartz model by using parameter estimators which is obtained from the Maximum Likelihood procedure is not really good enough in approximating the real data. It can be caused by several outliers contained in the real data which is used to estimate the parameter of the Brennan-Schwartz model. Moreover, the market prices of risk which are used to approximate the real data are based on trial and error. Therefore, any inaccuracy in our approximation to the real data stemmed from the imperfection of our parameter estimation.

# References

Ahn, S. and Shrestha, K. (2009), Estimation of Market Risk Premium for Japan, *Enterprise Risk Management*, 1, 33-43.

Aling, P. and Hassan, S. (2012), No-Arbitrage One-Factor Models of the South African Term-Structure of Interest Rates, *South African Reserve Bank Working Paper*.

Allen, E. (2007), *Modeling with Ito Stochastic Differential Equations*, Netherlands: Springer.

Arnold, L. (1974), *Stochastic Differential Equations*, New York: John Wiley & Sons.

Brennan, M.J. and Schwartz, E.S. (1980), Analyzing Convertible Bonds, *Journal of Financial and Quantitative Analysis*, 15, 907-929.

Brigo, D. and Mercurio, F. (2006), *Interest Rate Model - Theory and Practice: With Smile, Inflation, and Credit*, New York: Springer.

Handhika, T. (2011), Stability Analysis of the Brennan-Schwartz Model with Implementation, *Thesis*, University of Indonesia at Depok.

Hull, J.C. (2003), *Options, Futures, and Other Derivatives*, London: Prentice Hall.

Kloeden, P.E., and Platen, E. (1992), *Numerical Solution of Stochastic Differential Equations*, Heidelberg: Springer-Verlag.

Lo, A.W. (1988), Maximum Likelihood Estimation of Generalized Ito Processes with Discretely Sampled Data, *Econometric Theory*, Cambridge University Press, 4, 231-247.

Mikosch, T. (1998), *Elementary Stochastic Calculus with Finance in View*, Singapore: World Scientific.

Murni and Handhika, T. (2011), European Call Option Pricing On A Zero-Coupon Bond With The Brennan-Schwartz Model As A Short-Rate Model, *International Seminar on the Current Research Progress in Science and Technology (ISSTECH) 2011*, The Third International Joint Seminar UI – UKM.

Öksendal, B. (1998), *Stochastic Differential Equations: An Introduction with Applications*, New York: Springer.

Pennacchi, G. (2008), *Theory of Asset Pricing*, New York: Pearson.

Phillips, P.C.B. and Yu, J. (2009), Maximum Likelihood and Gaussian Estimation of Continuous Time Models in Finance, *Handbook of Financial Time Series*, Springer, 1597, 497-530.

Rebonato, R. (1998), *Interest-Rate Option Models 2nd Ed.: Understanding, Analyzing, and Using Models for Exotic Interest-Rate Options*, Chichester: John Wiley & Sons.

Rouah, F.D. and Vainberg, G. (2007), *Option Pricing Models and Volatility Using Excel-VBA*, New Jersey: John Wiley & Sons.

Shreve, S.E. (2004), *Stochastic Calculus for Finance II Continuous-Time Models*, New York: Springer.

Zeytun, S. and Gupta, A. (2007), A Comparative Study of the Vasicek and the CIR Model of the Short Rate, *ITWM Technical Report*, http://www.itwm.fraunhofer.de.