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Estimating Structural Models of Corporate Bond Prices in Indonesian Corporations

Lenny Suardi* and M Syamsudin**

This paper applies the maximum likelihood (ML) approaches to implementing the structural model of corporate bond, as suggested by Li and Wong (2008), in Indonesian corporations. Two structural models, extended Merton and Longstaff & Schwartz (LS) models, are used in determining these prices, yields, yield spreads and probabilities of default. ML estimation is used to determine the volatility of firm value. Since firm value is unobserved variable, Duan (1994) suggested that the first step of ML estimation is to derive the likelihood function for equity as the option on the firm value. The second step is to find parameters such as the drift and volatility of firm value, that maximizing this function. The firm value itself is extracted by equating the pricing formula to the observed equity prices. Equity, total liabilities, bond prices data and the firm's parameters (firm value, volatility of firm value, and default barrier) are substituted to extended Merton and LS bond pricing formula in order to valuate the corporate bond.

These models are implemented to a sample of 24 bond prices in Indonesian corporation during period of 2001-2005, based on criteria of Eom, Helwege and Huang (2004). The equity and bond prices data were obtained from Indonesia Stock Exchange for firms that issued equity and provided regular financial statement within this period. The result shows that both models, in average, underestimate the bond prices and overestimate the yields and yield spread.

Keywords: *structural models, corporate bond valuation, maximum likelihood estimation*

Introduction

Merton (1974) wrote a seminal paper that elaborated some of the implications of the paper by Black and Scholes (1973) for the pricing of corporate debt. This model expresses corporate debt and equity as option on the fundamental value or asset value of the firm. However this simple construction is inadequate to describe actual situations because it excludes the possibility of default before maturity, the effect of stochastic interest rates and the valuation of coupon-bearing bonds.

Many extensions of this model followed. This family model is known as

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structural model. Black and Cox (1976) introduced the default barrier which enabling default happens before maturity. Longstaff and Schwartz (1995) developed a simple framework that incorporates default barrier and stochastic interest rates to price corporate coupon bonds.

Eom, Helwege and Huang (2004) conducted a comprehensive empirical study of structural model. They obtained a sample of bonds from 1986-1997 in US corporation. To implement the structural models, they proxy the market value of corporate assets by the sum of the market value of equitie and the book value of total liabilities. Within this setting, Merton model overestimated the bond price whereas the Longstaff and Schwartz model tended to underestimated bond prices on average. Merton model underestimated yield spread on average, while Longstaff and Schwartz model tended to overestimate yield spread on average. EHH also proposed an extended Merton for coupon bonds and set some criteria for bond selection. Ronn and Verma (1986) proposed an answer to find the asset value volatility. They suggested solving a set of two equation relating the observed price of equity and estimated equity volatility to asset value and asset value volatility.

Duan (1994) derived a maximum likelihood (ML) approach. Li and Wong (2008) showed that structural models perform much better if the ML approach is used to estimate the model parameters. Based on this paper, the Merton model overestimates short-term bond yields but underestimates medium-term and long-term bond yields. The LS model demonstrated that its predictive power for medium-term is better than Merton.

This paper is conducted to valuate the corporate bond prices, yield, yield spread and default probability empirically, as suggested by Li and Wong, using data in Indonesian corporation between 2001 to 2005. We restrict our model to extended Merton and Longstaff and Schwartz models. The firm parameter is estimated by maximum likelihood estimation. Using the result of this estimation, this paper shows that the extended Merton model underestimated the bond prices and underestimates this yield. The same result is shown by LS model. By comparing these two models, extended Merton model tends to has higher estimation than LS model.

The rest of the paper is organized as follows. Section 2 reviews the implementation of two structural models with maximum likelihood approach. Section 3 reports our empirical framework and the result of valuation for extended Merton and LS models. Conclusions are presented in section 4.

Literature Review

Maximum Likelihood Estimation of Structural Model

This estimation empirically suggested by Li and Wong (2008) using bond prices data in US. The idea of maximum likelihood (ML) estimation proposed by Duan (1994) is to derive the likelihood function for the equity returns based on the assumption that the firm value is log-normally distributed and the equity value is an option on the firm. By maximizing the likelihood functions, parameter, such as the drift and volatility of a firm, are obtained. The firm asset value is then extracted by equating the pricing formula to the observed equity prices. We used the result of Merton (1974) and Brockman and Turtle (2003) that showed equity as European call option of firm value and down out call option of firm value, respectively.

Let $h(V,X,K;\sigma)$ is an equity pricing formula and $L(\mu,\sigma)$ is a likelihood function, where V,X,K,μ and σ is firm value, total liabilities, default barrier, drift parameter, and volatility parameter respectively. The implementation steps of MLE estimation are summarized by Li and Wong as follows:

1. Obtain parameters μ and σ by maximizing the likelihood function, $L(\mu,\sigma)$ subject to the constraint that $S_i = h(Vi,X,K;\sigma_V)$ for $i=1,2,...,n+1$ (1) where S_i is time series of observed equity price.
2. Calculate the firm's asset value by solving the above equation.
3. Substitute V_i and σ to the corporate bond pricing formula.

The likelihood function can change across model. This function, equity pricing formula and bond pricing formula are given in Appendix B.1, A.1, and A.2.

Data and Methodology

Criteria of Bond Selection

Based on criteria of Eom, Huang and Helwege (2004), we select a sample of firms with simple capital structures that have bonds with reliable prices and sufficient equity data. This limitation is desirable because with complicated capital structures raise doubt as to whether pricing error related to deficiencies of the model or to the facts that the model does not attempt to price the debt of firms with very complicated liabilities. Based on this limitation, we consider firms with only one or two public bonds and sinkable and subordinated bonds are excluded.

We use bond prices on last trading day of each December for period 2001-2007. These prices were obtained from the Surabaya Stock Exchange (now Indonesian Stock Exchange) database with time to maturity not less than one year on that date. There 147 bonds that meet these criteria. We restrict our sample to non-financial firms, so that the leverage ratios are comparable across firms. The bonds under consideration have fixed rate coupons and principal at maturity. In addition, we eliminate bonds from firms of public utility, as the return on equity revenues and the risk of default, are strongly influenced by regulators. There 45 bonds meet these requirements.

Moreover, we must have firms with publicly traded stock and provided regular financial statements in order to measure the market value of corporate asset and estimate its volatility. Therefore, we downloaded the market values of equities and total liabilities from database of library of Faculty of Economics, University of Indonesia for the period 2001-2005. At this stage, there are 27 bonds in the sample.

Lastly, by matching all available data with risk-free bond, Surat Utang Negara (SUN), at the day of observation, our sample ultimately contains 24 bonds issued by 22 firms. This restriction occurs because we need to estimate parameter of interest rate using yield of risk-free bond.

Table 1 presents summary statistics on the bonds and issuers in the sample.

Bond Specific Parameters

One of input parameter to determine bond prices is recovery rate (ω). We use the average value of this rate from historical data from IBRA that is 30% of corporate debt when the restructuring happened. The rebate rate is 30% based on paper Wibowo (2007). The last specific parameter is default barrier. A suggested by Li and Wong (2008), the default barrier for extended Merton and LS, respectively, are zero and 73.8% of total liabilities.

Parameter Estimation

Extended Merton (2003) and Longstaff and Schwartz (1995) model have a set of parameter that must be estimated. Parameters related to risk-free interest rate (r), the firm's initial value (V_0), and asset return volatility (σ), for each firm. The estimation result is obtained from Matlab software.

Risk-free Interest Rate Parameter

From Eom, Helwege, and Huang (2003), risk-free interest rate parameter for any particular day is estimated based on constant maturity SUN yield data on that day. The estimated rate for both models, from appendinx B.2 , are presented in Table 2.

Firm Value and Asset Return Volatility Parameter

The average value of estimate of firm value and its volatility with maximum

Table 1. Summary Statistics of the Bonds

Panel A.

Characteristic	Mean	Standard Deviation	Minimum	Maximum
Time to Maturity (T)	4.39	0.68	2.58	5.50
Coupon Rate (c)	13.388	0.992	10.750	15.125
Yield to Maturity (y)	0.136	0.015	0.106	0.167
Market Capitalization (thousand) (S)	8517857428	16907755629	91080000	77662500000
Total Liabilities (thousand) (X)	3834811579	5059510397	117622130	18471805000

Panel B.

Year	# of bonds	T	c	y	S (000)	X (000)
2003	11	4.61	13.81	0.1384	8890872923	3084518215
2004	11	3.88	13.16	0.1296	6850985989	3453935140
2005	2	4.92	12.13	0.1635	15634065125	10056245490

Panel A shows that our sample contains bonds with maturities that range 2.58 years to 5.5 years. This narrow range of maturities prohibits us to study the effect of longer time to maturity. The coupon rate, averagely, is 13.38% which is paid annually for all bonds. The range of yield to maturity is relatively narrow, from 10.6% to 16.7%. Our sample includes of different sizes of firms that carry at least Rp91.08 billions of market capitalization to a maximum IDR. 77.66 trillions. Total liabilities of the firms range from IDR.117 billions to IDR.18.47 trillions. Panel B presents the mean of the maturity time, coupon rate, yield to maturity, market capitalization, and total liabilities of each year. The year of observation is reduced to 2003-2005 since we can not get the yield of risk-free bonds (SUN) for year 2001 and 2002 with time to maturity less than 7 years.

Table 2. Risk Free Interest Rate Estimate

Year	T	Nelson Siegel	Vasicek					
		r	α	β	η	r	D(0,T)	rho
2003	4.58	0.1255	0.0165	0.0969	0.0562	0.1213	0.5699	-0.0491
	4.83	0.1254	0.0151	0.0996	0.0581	0.124	0.5588	-0.0933
	4.42	0.1256	0.0167	0.0882	0.0608	0.1216	0.5778	0.0017
	4.50	0.1256	0.0182	0.0989	0.0647	0.121	0.5736	0.0625
2004	2.58	0.1066	0.0204	0.098	0.109	0.1014	0.7666	-0.1372
	3.00	0.1072	0.0212	0.098	0.1105	0.1026	0.7342	-0.0484
	5.00	0.1086	0.0176	0.0999	0.0965	0.1196	0.5930	0.0345
	4.83	0.1086	0.0182	0.0929	0.0989	0.1164	0.6040	-0.0665
	4.42	0.1083	0.0186	0.0964	0.0974	0.1115	0.6320	-0.1589
	4.58	0.1084	0.0185	0.0932	0.1099	0.1207	0.6206	-0.1112
	4.50	0.1084	0.0206	0.1064	0.1101	0.1174	0.6263	-0.0482
	2.92	0.1071	0.0204	0.1019	0.1436	0.115	0.7405	-0.0215
	3.92	0.108	0.0223	0.0848	0.1098	0.1039	0.6670	0.0275
	2.33	0.1061	0.0216	0.0976	0.109	0.099	0.7867	-0.0557
2005	5.50	0.1362	0.0331	0.0938	0.1053	0.1244	0.4741	-0.1713
	4.33	0.1365	0.0346	0.997	0.103	0.1152	0.5632	-0.0781

The Nelson Siegel estimate is used for extended Merton model while the Vasicek estimate is used for Longstaff and Schwartz.

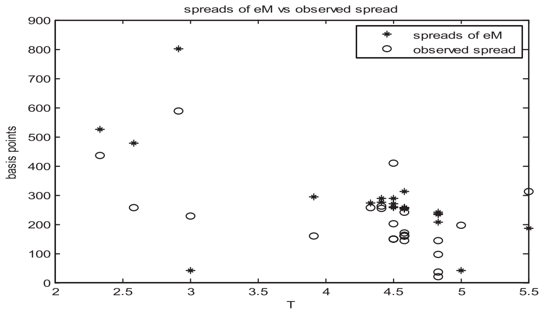
Table 3. Estimate Parameters

	firm value	volatility
Merton	10692360929.422 (18840703283.362)	0.330 (0.146)
Ls	10719755916.205 (18835083807.954)	0.283 (0.130)

Table 4. Empirical Results

Bond Pricing Formula	Pricing Errors	Errors in yield	Errors in Spread
	mean (std. dev)	mean (std. dev)	mean (std. dev)
extended Merton	-2.54%	5.11%	95.62%
	(4.70%)	(8.61%)	(224.71%)
LS	-10.94%	19.59%	236.16%
	(5.050%)	(9.59%)	(358.67%)

Figure 1. Spreads Of eM vs Observed Spreads



likelihood estimation is presented in Table 3. The Merton model attempts to estimate firm value volatility higher than the LS model averagely, but it estimates firm value lower than LS model. The values in the bracket indicate the averages of standard deviation of estimate parameter.

In this study, as shown by Duan (1994) and Li and Wong (2008), maximum likelihood estimation leads to an upward bias in asset drift, but that the other parameters are obtained with a high quality. However, this bias has no impact on testing of corporate bond pricing model. Structural models value corporate bonds in the risk neutral world in which asset drift is replace by the risk-free interest rate. Thus, the biased drift value plays no role in corporate bond pricing formulas. The inclusion of the drift in the estimation procedure aims to enhance the estimation quality of the volatility.

Result and Discussion

Empirical Results

In this part we discuss the ability of the models to fit market prices, yields, yield spreads and default probability. We present the percentage pricing errors, the percentage errors in yields, and the percentage errors in yield spreads. The percentage errors are calculated as predicted value minus the observed value divided by the observed value.

Extended Merton (eM) Model

The empirical result for extended Merton model is shown is first row of Table 4. The results suggest that this model less underestimates the bond prices on average. There is different result comparing to the past researches about the performance of extended Merton model. Those researches shows that Merton model attempted to overestimate the bond prices.

The average error of yields and yields spread of this model is positive. The plot of predicted bond spreads and the actual bond spreads against maturity is given in Figure 1.

Figure 1 shows that extended Merton overestimates the spreads. Unfortunately, we can not capture the effect of difference value of time to maturity because the narrow range of this value. The estimates for this model are based on risk-free rates from Nelson Siegel model. Using the Vasicek estimates, does not give a significance difference in results.

Longstaff and Schwartz (LS) Model

The second row of Table 4 summarizes the result from implementing the LS model. The LS model has lower predicted prices than observed prices and predicted prices of extended Merton model. On the other hand, this model has much higher predicted spreads than extended Merton. The same results is shown by Eom, Helwege and Huang (2004) and Li and Wong (2004). In this sample, error in yield spreads an average more than 236%. Range in spreads is extremely high, reaches 864 of basis point; because there are 3 observation having spreads more than 900 basis points, which have lowest time to maturity:

The LS model predicts default probability higher than extended Merton model. The plot is shown below.

Conclusion

This paper empirically test the extended Merton (2004) and Longstaff and Schwartz (1995) as corporate bond pricing models

Figure 2. Spreads of LS Vs Observed Spreads

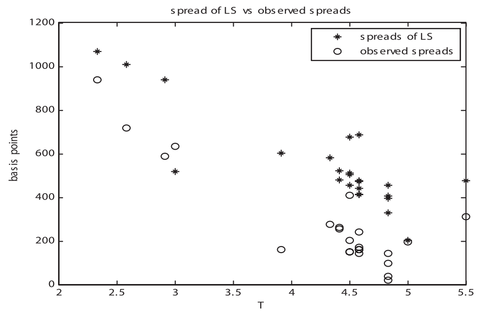
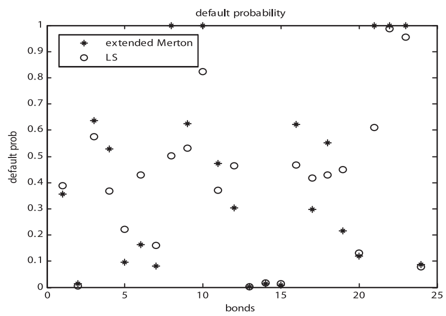


Figure 3. Default Probability eM vs LS



using a sample of bonds belonging to firms with simple capital structure in Indonesian corporation between 2001-2005. The parameters of the firm (firm value and its volatility) are estimated with maximum likelihood estimation, as suggested by Li and Wong (2008). On average, the maximum likelihood estimation of firm value with Merton model tends to be lower than the estimate from the Longstaff and Schwartz model. On the other hand, the estimated volatility of firm value by the Merton model is higher in general than the estimation by the Longstaff and Schwarz model.

Both models underestimate the price of corporate bonds, averagely. It implies that both models overestimate the yield of corporate bonds. If we compare the prices of bonds for these two models, the extended Merton predicts a higher price than LS models. On the other hand, the extended Merton predicts a lower yield than LS model. The LS model extremely overestimates the yield spreads of bonds comparing to the result of extended Merton model. Averagely, default probability of LS model is higher than extended Merton model.

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Appendix

A. Formulas for Prices of Bonds

A.1. Original Merton Model

The original Merton model considers a corporate zero coupon bonds with a maturity T and face value X. The model assumes a constant interest rates r and market values of assets V follow a geometric Brownian motion, i.e.

$$dV=\mu Vdt+\sigma VdZ_1, \tag{2}$$

where μ and σ is the drift and volatility of market values of assets respectively and is a standard Brownian motion.

Assuming no intermediate default, the payoff of the bond is the minimum of the face amount of the bond and the market value of assets at maturity V_T The equity pricing formula at t=0 is expressed as

$$S=h(V,X;\sigma_V)=VN(d_1)-Xe^{(-rT)}N(d_2), \tag{3}$$

and the corporate bond prices can be written as:

$$F^M=(V_0,X,T)=Xe^{(-rT)}N(d_2)+VN(-d_1), \tag{4}$$

where

$$d_1=\frac{\ln\left(\frac{V_0}{X}\right)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \tag{5}$$

$$d_2=\frac{\ln\left(\frac{V_0}{X}\right)+\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \tag{6}$$

and N(•)= cumulative distribution of standard normal random variable.

A.2 Extended Merton Model

The extended Merton Model proposed by Eom, Helwege, and Huang (2004) to treat a coupon-bearing bond as portfolio of zero coupon bonds. Consider a defaultable bond with maturity T and unit face value that pays annual coupons at an annual rate of c. Let be the nth coupon date.

Let the default barrier for each coupon dates are equal to *K* and default is triggered if asset value is below to *K* on coupon dates. The price of coupon bond is equal to portfolio of zeros and can be written as follows:

$$F^{eM} = \sum_{i=1}^{T-1}\left(Xce^{-rT_i} B\left(d_2\left(K,T_i\right)\right) + \omega Xce^{-rT_i}\begin{bmatrix} N\left(d_2\left(\omega Xc,T_i\right)\right) - N\left(d_2\left(K,T_i\right)\right) \\ +V_0 N\left(-d_1\left(\omega Xc,T_i\right)\right)\end{bmatrix}\right) \\ + X\left(1+c\right)e^{-rT} N\left(d_2\left(K,T\right)\right) + V_0 N\left(-d_1\left(\omega X\left(1+c\right),T\right)\right) \\ + \omega X\left(1+c\right)e^{-rT}\left[N\left(d_2\left(\omega X\left(1+c\right),T\right)\right) - N\left(d_2\left(K,T\right)\right)\right], \quad (7)$$

where ω is recovery rate.

A.3 Longstaff and Schwartz Model

For the LS model, asset prices are assumed to follow equation (2) and interest rates are assumed to be stochastic with dynamics of is written as

$$dr=(\zeta-\beta r)dt+\eta dZ_2, \quad (8)$$

where ζ,β,η are constant and dZ_2 is standard Brownian motion. This model is modified form of Ornstein-Uhlenbeck process as the specific case of Vasicek model (1977):

$$dr=\kappa(\theta-r)dt+\eta dZ_2, \quad (9)$$

where θ is the long-run average of instantaneous spot rate, κ reflects the speed of mean reversion and η is the volatility parameter of process. The instantaneous correlation of dZ1 and dZ2 is ρdt and the risk premium λ is assumed to be constant.

Under the LS framework, default occurs if the market value of asset at time , reaches the threshold value *K*, or equivalent $L=V/K$ reaches ones for the first time. Let , the pricing formula for a corporate coupon bond, based on portfolio of zeros, can be calculated as

$$F^{LS}\left(X,r_0,T\right) = Xc\sum_{i=1}^{T-1} D\left(r,T_i\right) - \left[w_l D\left(r,T_i\right)Q\left(L,r,T_i\right)\right] \\ + X\left(1+c\right)D\left(r,T\right) - \omega_l X\left(1+c\right)D\left(r,T\right)Q\left(L,r,T\right), \quad (10)$$

where

$$Q(L,r,T,n) = \sum_{i=1}^{n} q_i,$$

$$q_1 = N(a_1),$$

$$q_i = N(a_i) - \sum_{j=1}^{i-1} q_j N(b_{ij}),\ i=1,2,\ldots,n, \quad (11)$$

$$a_i = \frac{\left(-\ln L - M\left(iT\ \ n,T\right)\right)}{\sqrt{S\left(iT\ \ n\right)}},$$

$$\begin{aligned} M(t,T) \;=\; & \left(\frac{\alpha - \rho\sigma\eta}{\beta} - \frac{\eta^2}{\beta^2} - \frac{\sigma^2}{2} \right) t \\ & + \left(\frac{\rho\sigma\eta}{\beta^2} + \frac{\eta^2}{2\beta^3} \right) \exp(-\beta T)\left(\exp(\beta t) - 1\right) \\ & + \left(\frac{r}{\beta} - \frac{\alpha}{\beta^2} - \frac{\eta^2}{\beta^3} \right) \left(1 - \exp(-\beta t)\right) \\ & + \left(\frac{\eta^2}{2\beta^2} \right) \exp(-\beta t)\left(1 - \exp(-\beta t)\right), \end{aligned} \tag{12}$$

$$\begin{aligned} S(t) \;=\; & \left(\frac{\rho\sigma\eta}{\beta} + \frac{\eta^2}{\beta^3} + \sigma^2 \right) t \\ & - \left(\frac{\rho\sigma\eta}{\beta^2} + \frac{2\eta^2}{\beta^3} \right) \left(1 - \exp(-\beta t)\right) \\ & + \left(\frac{\eta^2}{2\beta^3} \right) \left(1 - \exp(-2\beta t)\right) \\ & + \left(\frac{\eta^2}{2\beta^2} \right) \exp(-\beta t)\left(1 - \exp(-\beta t)\right), \end{aligned} \tag{13}$$

and

$$Q(L,r,T) \;=\; \lim_{n\to\infty} Q(L,r,T,n), \tag{14}$$

$N(.)$ is cumulative distribution function of normal standard.

The down out call option as equity pricing, based on Brockman and Turtle (2003), can be written as

$$\begin{aligned} S \;&=\; h(V_i, X, K; \sigma_V) = DOC(V, X, K, R) \\ &=\; V\left(N(a) - \left(K/V\right)^{2\xi} N(b) \right) \\ &\quad - Xe^{-rT}\left(N\left(a - \sigma\sqrt{T}\right) - \left(K/V\right)^{2\xi-2} N\left(b - a - \sigma\sqrt{T}\right) \right) \\ &\quad + R\left(\left(K/V\right)^{2\xi-1} N(c) + \left(V/K\right) N\left(c - 2\xi\sigma\sqrt{T}\right) \right), \end{aligned} \tag{15}$$

where V is the market value of firm assets, X total liabilities, K is default barrier, σ is asset volatility, r is risk-free interest rates, T is the time to maturity, R is the rebate paid to the equity holders upon default, $N(.)$ is cumulative distribution function of normal standard random variable and

$$\xi = \frac{r}{\sigma^2} + \frac{1}{2},$$

$$a = \begin{cases} \dfrac{\ln(V/X) + \left(r + \sigma^2/2\right)T}{\sigma\sqrt{T}}, & for\ X \geq K, \\[2ex] \dfrac{\ln(V/K) + \left(r + \sigma^2/2\right)T}{\sigma\sqrt{T}}, & for\ X < K, \end{cases} \tag{16}$$

$$b = \begin{cases} \dfrac{\ln\left(K^2/VX\right)+\left(r+\sigma^2/2\right)T}{\sigma\sqrt{T}}, \mathit{for}\ X \geq K, \\ \dfrac{\ln\left(K/V\right)+\left(r+\sigma^2/2\right)T}{\sigma\sqrt{T}}, \mathit{for}\ X < K. \end{cases} \tag{17}$$

B. Parameter Estimation

B.1 Likelihood Function

From equation (2) we get

$$\ln\left(V_t\right) = \ln\left(V_0\right) + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\ dZ_1, \tag{18}$$

where V_t is the market value of asset at time t, μ is the drift parameter, σ is the asset volatility, and dZ_1 is a standard Brownian motion. Let interval $\ln(V_i)=v_i$ and $\ln(V_{i-1})=v_{i-1}$, $i=1,2,...,n$, for $0<t<s<T$, based on Li and Wong (2008), the density function of v_i given v_{i-1} for Merton model is expressed by:

$$g\left(v_1 | v_{i-1}\right) = \frac{1}{\sigma\sqrt{2\pi\left(t_i - t_{i-1}\right)}} \exp\left\{ -\frac{\left[v_i - v_{i-1} - \left(\mu - \frac{\sigma^2}{2}\right)\left(t_i - t_{i-1}\right)\right]^2}{2\sigma^2\left(t_i - t_{i-1}\right)} \right\}. \tag{19}$$

and the density function of barrier dependent model is expressed by:

$$g\left(v_1 | v_{i-1}\right) = \frac{1}{\sigma\sqrt{2\pi\left(t_i - t_{i-1}\right)}} \exp\left\{ -\frac{\left[v_i - v_{i-1} - \left(\mu - \frac{\sigma^2}{2}\right)\left(t_i - t_{i-1}\right)\right]^2}{2\sigma^2\left(t_i - t_{i-1}\right)} \right\} - e^{2(\zeta-1)(b-v_{i-1})} \frac{1}{\sigma\sqrt{2\pi\left(t_i - t_{i-1}\right)}} \exp\left\{ -\frac{\left[v_i - v_{i-1} - \left(\mu - \frac{\sigma^2}{2}\right)\left(t_i - t_{i-1}\right) - 2b\right]^2}{2\sigma^2\left(t_i - t_{i-1}\right)} \right\}, \tag{20}$$

where

$$b = \ln K \text{ and } \xi = \frac{r}{\sigma^2} + \frac{1}{2}$$

Using the equity pricing formula in (1), the log-likelihood of μ and σ with Merton model and barrier dependent model is written as:

$$L(\mu,\sigma) = \sum_{i=2}^{n} \ln f\left(S_i \middle| S_{i-1}, \mu,\sigma\right), S_i = S(t_i) \qquad (21)$$

where f denotes the probability density function of S, and $S(t_i)$ denotes the market value of equity at time t_i. After applying the standard change of variable technique, the likelihood function is obtained as follows:

$$L(\mu,\sigma) = \sum_{i=2}^{n} \left\{\ln g\left(v_i \middle| v_{i-1}\right) - \ln\left[V_i.\Delta(V_i)\Big|_{V=V_i}\right]\right\} \qquad (22)$$

where option delta, $\Delta(V_i)$, is calculated by differentiating the equity pricing formula $h(V_i,X,K;\sigma)$ with respect to V.

We then estimate the parameters by solving the following optimization problem:

$$\max_{\mu,\sigma} L(\mu,\sigma) \text{ subject to } S(t_i) = h(V_i, X, K;\sigma), i = 1, 2, \ldots, n. \qquad (23)$$

B.2 Interest Rate Parameter

Let instantaneous forward rate at time to maturity *T-t* denoting by $r(T-t)$. The Nelson Siegel model (1987) for yield to maturity, $y(t,T,\Theta_r)$ as average of forward rates is given by:

$$y(t,T;\Theta_r) = \beta_0 + \delta\left(\beta_1 + \beta_2\right)\frac{\left(1 - e^{(-(T-t)/\delta)}\right)}{T-t} - \beta_2 e^{(-(T-t)/\delta)} \qquad (24)$$

where $\Theta_r=(\delta,\beta_0,\beta_1,\beta_2)$, and δ and β_0 need to be positive. To fit the model to constant maturity treasury rates on day t, one choose parameter in Θ_r to minimize the sum of error squared between the model yield and the yield of constant maturity treasury.

In the Vasicek (1977) model, let $\Theta_r=(q,m,\upsilon,\lambda)$ denotes the set of parameter. Risk-free zero coupon prices with unit face value and time to maturity T is given by:

$$D_t = (T,\Theta_r) = A(T;\Theta_r)\exp\left[-B(T;\Theta_r)r_t\right] \qquad (25)$$

where

$$A(T;\Theta_r) \equiv \exp\left[\phi B(T;\Theta_r) - T - \frac{\upsilon^2 B^2(T;\Theta_r)}{4q}\right], \qquad (26)$$

$$B(T;\Theta_r) \equiv \frac{1}{q}\left[1 - \exp(-qT)\right], \text{ and } \phi \equiv \mathrm{m} + \frac{\upsilon\lambda}{q} - \frac{\upsilon^2}{2q^2}. \qquad (27)$$

The set of parameter $\Theta_r=(q,m,\upsilon,\lambda)$ can be estimated with maximum likelihood estimation.